Proof systems for the logics for social behaviour

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Logics for social behaviour

Logics aimed at capturing agency, social structure, information flow...

These logics come in **large families**:

- many similarities, but also many idiosyncratic differences.

They have proven **impervious** to the standard proof-theoretic treatment

- hurdles due to the very features which make them successful at capturing **real life**;
  - lack of closure under uniform substitution, extralinguistic labels, etc

Our contribution: *typed calculi*.
Case studies: DEL, PDL, Logic of Resources and Capabilities, Inquisitive logic, Linear Logic, Lattice logic.
Near future: Game logic, Facebook logic, PDEL, dependence / independence logic...

‘Good’ Proof Systems: Desiderata

- **Independence**:
  - suitability for proof-theoretic semantics
- **Modularity**:
  - rules clearly related to pre-existing axioms
  - space of LSB charted by adding/removing rules
  - transfer of results with minimal changes
- **Good mathematical properties**:
  - soundness & completeness
  - cut-elimination & sub-formula property
    - $\rightsquigarrow$ analyticity, decidability
  - conservativity

Display Calculi

- Natural generalization of Gentzen's sequent calculi;
- sequents $X \vdash Y$, where X and Y are STRUCTURES:
  - formulas are **atomic structures**
  - built-up: **structural connectives** (generalizing meta-linguistic comma in sequents $\phi_1, \ldots, \phi_n \vdash \psi_1, \ldots, \psi_m$)
  - generation **trees** (generalizing sets, multisets, sequences)
- DISPLAY PROPERTY: syntactic counterpart of **adjunction**
- Canonical proof of cut elimination

Cut elimination [Belnap 82, Wansing 98]

Theorem

Any **proper display calculus** enjoys cut elimination and subformula property.

Definition

A **proper display calculus** verifies all conditions in the following list:

1. structures can disappear, formulas are **forever**;
2. **tree-traceable** formula-occurrences, via suitably defined congruence:
   - same shape, same position, non-proliferation;
3. **principal = displayed**
4. rules are closed under **uniform substitution** of congruent parameters;
5. **reduction strategy** exists when both cut formulas are principal.

Cut elimination: Proof by induction

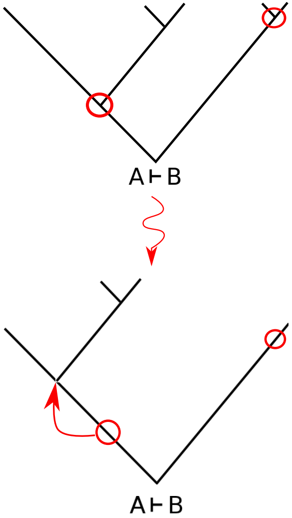
Complexity of the cut formula

$$\dfrac{\dfrac{\dfrac{\vdots\ \pi_1}{Z \vdash \circ A}}{Z \vdash \Box A} \qquad \dfrac{\dfrac{\vdots\ \pi_2}{A \vdash Y}}{\Box A \vdash \circ Y}}{Z \vdash \circ Y}\ \mathit{Cut}$$

$$\Downarrow$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots\ \pi_1}{Z \vdash \circ A}}{\bullet Z \vdash A}\ \text{Display} \qquad \dfrac{\vdots\ \pi_2}{A \vdash Y}}{\bullet Z \vdash Y}\ \mathit{Cut}}{Z \vdash \circ Y}\ \text{Display}$$

Height of the cut



Properly displayable logics

1. the logic of any **basic** normal distributive lattice expansion (DLE);
2. the logic of any basic **tense** DLE;
3. axiomatic extensions of these with so called **analytic inductive inequalities**.

Analytic inductive inequalities

- syntactically defined subclass of Sahlqvist(-type) inequalities;
- algebraic inspired definition, applying to any DLE-signature;
- equivalent proof-theoretic definition in [Ciabattoni Ramanayake 2015].

Is DEL properly displayable?

$$\langle\alpha\rangle p \leftrightarrow Pre(\alpha) \wedge p$$

$$\langle\alpha\rangle(A \vee B) \leftrightarrow \langle\alpha\rangle A \vee \langle\alpha\rangle B$$

$$\langle\alpha\rangle\neg A \leftrightarrow Pre(\alpha) \wedge \neg\langle\alpha\rangle A$$

$$\langle\alpha\rangle\langle \mathtt{a}\rangle A \leftrightarrow Pre(\alpha) \wedge \bigvee\{\langle \mathtt{a}\rangle\langle\beta\rangle A \mid \alpha\mathtt{a}\beta\}$$

Is PDL properly displayable?

$$[\alpha]\,(A \to B) \to ([\alpha]\,A \to [\alpha]\,B)$$

$$[\alpha \cup \beta]\,A \leftrightarrow [\alpha]\,A \wedge [\beta]\,A$$

$$[\alpha\,;\,\beta]\,A \leftrightarrow [\alpha][\beta]\,A$$

$$[?A]\,B \leftrightarrow (A \to B)$$

$$[\alpha]\,(A \wedge B) \leftrightarrow [\alpha]\,A \wedge [\alpha]\,B$$

$$[\alpha^*]\,A \leftrightarrow A \wedge [\alpha]\,[\alpha^*]\,A$$

$$A \wedge [\alpha^*]\,(A \to [\alpha]\,A) \to [\alpha^*]\,A$$

Are Inquisitive Logic/Dependence logic properly displayable?

$(\chi \to (\phi \vee \psi)) \to (\chi \to \phi) \vee (\chi \to \psi)$ with χ classical (restricted)

$\neg\neg\chi \to \chi$ whenever χ classical (restricted)

Is Linear Logic properly displayable?

$$\frac{Y \vdash A}{Y \vdash !A} \qquad \frac{A \vdash X}{!A \vdash X}$$

$$\frac{X \vdash A}{X \vdash ?A} \qquad \frac{A \vdash Z}{?A \vdash Z}$$

$!!A = !A$

$!A \leq A$

$A \vdash B$ implies $!A \vdash !B$

$!\top = 1$

$!(A \& B) = !A \otimes !B$

Is Lattice Logic properly displayable?

$$\frac{X \vdash A \qquad X \vdash B}{X \vdash A \wedge B} \qquad \frac{A \vdash X}{A \wedge B \vdash X} \qquad \frac{B \vdash X}{A \wedge B \vdash X}$$

$$\frac{A \vdash X \qquad B \vdash X}{A \vee B \vdash X} \qquad \frac{X \vdash A}{X \vdash A \vee B} \qquad \frac{X \vdash A}{X \vdash A \vee B}$$

In general lattices, $\wedge$ and $\vee$ are adjoins but not residuals.
Belnap's approach: no structural counterparts.
Hence: no structural rules capturing interaction between $\wedge$ and $\vee$ and other connectives...

Analysis

Issues of different types

1. failure of closure under substitution;
2. labels, extra-linguistic abbreviations;
3. absence of semantically justified residuals;
4. subtler violations of analyticity

Drastic solutions

1. second-guess the presentations of logical systems: which information is key?
2. algebraic/order-theoretic analysis
   - alternative, more amenable semantic frameworks?
   - natural adjunction situations?
3. semantic motivation for richer languages
4. embedding old language into new (or translating new into old)

Diagnosis & cure

$$\textit{swap-out}_L \; \frac{\Big(Pre(\alpha)\,;\{\mathtt{a}\}\{\beta\}\,X \vdash Y \mid \alpha\mathtt{a}\beta\Big)}{Pre(\alpha)\,;\{\alpha\}\{\mathtt{a}\}X \vdash \mathbf{;}\Big(Y \mid \alpha\mathtt{a}\beta\Big)}$$

× Diagnosis:

- key interactions encoded by extralinguistic devices
- formulas encode information about parameters

√ Cure:

- add structural connectives
- add types

The multi-type approach

- Ag Act Fnc Fm;
  - no ancillary symbols; all types are **first-class citizens**;
- Additional expressivity:
  - operational connectives **merging different types** (à la Abramsky, Vickers):

$$
\begin{array}{lcll}
\vartriangle_1, \blacktriangle_1 & : & \mathsf{Act} \times \mathsf{Fm} \to \mathsf{Fm} & \langle\alpha\rangle A \rightsquigarrow \alpha \vartriangle_1 A \\
\vartriangle_2, \blacktriangle_2 & : & \mathsf{Ag} \times \mathsf{Fm} \to \mathsf{Fm} & \langle \mathsf{a}\rangle A \rightsquigarrow \mathsf{a} \vartriangle_2 A \\
\vartriangle_3, \blacktriangle_3 & : & \mathsf{Ag} \times \mathsf{Fnc} \to \mathsf{Act} & \alpha \mathsf{a} \beta \rightsquigarrow \mathsf{a} \vartriangle_3 \alpha
\end{array}
$$

- Modularity: Charting the space of DEL-type logics by adding or removing types (games, strategies, coalitions) and algebraic structure within each type.

  For $1 \leq i \leq 3$,

$\vartriangle_i$		$\blacktriangle_i$		$\vartriangleright_i$		$\blacktriangleright_i$	
$\vartriangle_i$		$\blacktriangle_i$			$-\!\!\vartriangleright_i$		$-\!\!\blacktriangleright_i$

A glimpse at rules for DEL

Single-type, first version: rules with side conditions & labels;

$$\textit{swap-out}_L \; \frac{\Big(\textit{Pre}(\alpha)\,;\{\mathtt{a}\}\{\beta\}\,X \vdash Y \mid \alpha\mathtt{a}\beta\Big)}{\textit{Pre}(\alpha)\,;\{\alpha\}\{\mathtt{a}\}X \vdash \boldsymbol{;}\Big(Y \mid \alpha\mathtt{a}\beta\Big)}$$

Single-type, emended: purely structural, but labels still there;

$$\textit{swap-out}'_L \; \frac{\Big(\Phi_\alpha\,;\{\mathtt{a}\}\{\beta\}\,X \vdash Y \mid \alpha\mathtt{a}\beta\Big)}{\Phi_\alpha\,;\{\alpha\}\{\mathtt{a}\}X \vdash \boldsymbol{;}\Big(Y \mid \alpha\mathtt{a}\beta\Big)}$$

Multi-type: no side conditions and no labels.

$$\text{swap-out}_L \; \frac{(\mathtt{a} \blacktriangle F) \blacktriangle (\mathtt{a} \blacktriangle X) \vdash Y}{\mathtt{a} \blacktriangle (F \blacktriangle X) \vdash Y}$$

$$\dfrac{\mathtt{a} \vdash \mathtt{a} \qquad \alpha \vdash \alpha}{\mathtt{a} \blacktriangle \alpha \vdash \mathtt{a} \blacktriangle \alpha}$$

$$\dfrac{\mathtt{a} \blacktriangle \alpha \vdash \mathtt{a} \blacktriangle \alpha \qquad A \vdash A}{(\mathtt{a} \blacktriangle \alpha) \triangle A \vdash (\mathtt{a} \blacktriangle \alpha) \triangle A}$$

$$\dfrac{\mathtt{a} \vdash \mathtt{a} \qquad (\mathtt{a} \blacktriangle \alpha) \triangle A \vdash (\mathtt{a} \blacktriangle \alpha) \triangle A}{\mathtt{a} \triangle ((\mathtt{a} \blacktriangle \alpha) \triangle A) \vdash \mathtt{a} \triangle ((\mathtt{a} \blacktriangle \alpha) \triangle A)}$$

$$\dfrac{\mathtt{a} \triangle ((\mathtt{a} \blacktriangle \alpha) \triangle A) \vdash \mathtt{a} \triangle ((\mathtt{a} \blacktriangle \alpha) \triangle A)}{(\mathtt{a} \blacktriangle \alpha) \triangle A \vdash \mathtt{a} \blacktriangleright (\mathtt{a} \triangle ((\mathtt{a} \blacktriangle \alpha) \triangle A))}$$

$$\dfrac{(\mathtt{a} \blacktriangle \alpha) \triangle A \vdash \mathtt{a} \blacktriangleright (\mathtt{a} \triangle ((\mathtt{a} \blacktriangle \alpha) \triangle A))}{A \vdash (\mathtt{a} \blacktriangle \alpha) \blacktriangleright (\mathtt{a} \blacktriangleright (\mathtt{a} \triangle ((\mathtt{a} \blacktriangle \alpha) \triangle A)))}$$

s-out

$$\dfrac{A \vdash (\mathtt{a} \blacktriangle \alpha) \blacktriangleright (\mathtt{a} \blacktriangleright (\mathtt{a} \triangle ((\mathtt{a} \blacktriangle \alpha) \triangle A)))}{A \vdash \mathtt{a} \blacktriangleright (\alpha \blacktriangleright (\mathtt{a} \triangle ((\mathtt{a} \blacktriangle \alpha) \triangle A)))}$$

$$\vdots$$

$$\dfrac{\alpha \triangle (\alpha \blacktriangle (\alpha \mathrel{-\!\!\rhd} (\mathtt{a} \triangle A))\,;\mathrm{I}) \vdash \mathtt{a} \triangle ((\mathtt{a} \blacktriangle \alpha) \triangle A)}{(\alpha \mathrel{-\!\!\rhd} (\mathtt{a} \triangle A))\,;(\alpha \triangle \mathrm{I}) \vdash \mathtt{a} \triangle ((\mathtt{a} \blacktriangle \alpha) \triangle A)}\ \textit{conj}$$

$$\vdots$$

$$\dfrac{\alpha \mathrel{-\!\!\rhd} (\mathtt{a} \triangle A) \vdash \alpha \triangle \top \rightarrow \mathtt{a} \triangle ((\mathtt{a} \blacktriangle \alpha) \triangle A)}{[\alpha]\langle \mathtt{a} \rangle A \vdash Pre(\alpha) \rightarrow \bigvee \left(\langle \mathtt{a} \rangle \langle \beta_i \rangle A \right)}$$

Linear logic: analysis

$!!a = !a$
$!a \leq a$
$a \leq b$ implies $!a \leq !b$
$!\top = 1$
$!(a \& b) = !a \otimes !b$

$! : \mathbb{L} \to \mathbb{L}$ interior operator. Let Range($\mathbb{L}$) := $\mathbb{B}$. Then $! = e \circ \iota$, where

$$e \dashv \iota : \mathbb{L} \to \mathbb{B}$$

and $e : \mathbb{B} \hookrightarrow \mathbb{L}$ embedding. Moreover, $\mathbb{B}$ can be endowed with the structure of a BA

$$\frac{Y \vdash (!)\Gamma}{EY \vdash \Gamma}$$

Problem: $!(a \& b) \leq !a \otimes !b$ non analytic.
Solution: ι surjective $\Rightarrow$ condition above semantically equivalent to

$$e(\alpha \wedge \beta) \leq e(\alpha) \otimes e(\beta)$$

which is analytic!

Canonical cut elimination, with types

Definition

A sequent $x \vdash y$ is ***type-uniform*** if x and y are of the same type.

Theorem (Canonical cut elimination)

If a calculus satisfies the properties below, then it enjoys cut elimination.

Canonical cut elimination, with types

1. structures can disappear, formulas are **forever**;
2. **tree-traceable** formula-occurrences, via suitably defined congruence:
   - same shape, same position, same type, non-proliferation;
3. **principal** = **displayed** (Exception: principal fma's in axioms)
   - Generaliz.: axioms are **closed** under display rules; every term introduced in display in one position **at least**.
4. rules are closed under **uniform substitution** of congruent parameters within each type;
5. **reduction strategy** exists when cut formulas are both principal.

Specific to multi-type calculi:

6. **type-uniformity** of derivable sequents;
7. rules preserve **type-uniformity**.

Conclusions

Display $\rightsquigarrow$ Multi-type display $\rightsquigarrow$ Multi-type display-type

- Giving names to problems
- Identifying requirements
- Uniform routes to soundness, completeness, cut-elimination, subformula property, conservativity
- macroscopic issues made smoother, scope of analiticity enlarged.
- Uniform route to decidability?

Next developments:

Logics, Decisions, and Interactions

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Canonical cut elimination

Two main cases + subcases.

(a) **Both cut formulas are principal.** by 5. (cut is either eliminated or "broken down" into cuts of lower rank).

(b) **At least one cut formula is parametric.**

- Subcase (b1): a_u principal in axiom. Then,

$$
\begin{array}{c}
(x' \vdash y')[a_u^{pre}, a_{suc}] \\[1em]
\dfrac{\begin{array}{c}\vdots\ \pi_1 \\ x \vdash a\end{array} \qquad \begin{array}{c}\vdots\ \pi_2 \\ a \vdash y\end{array}}{x \vdash y}
\end{array}
\quad \rightsquigarrow \quad
\begin{array}{c}
\dfrac{\begin{array}{c}\vdots\ \pi_1 \\ x \vdash a\end{array} \qquad a_u \vdash y''[a_{suc}]}{x \vdash y''[a_{suc}]} \\
\vdots\ \pi'' \\
(x' \vdash y')[x^{pre}, a_{suc}] \\
\vdots\ \pi_2[x/a_u] \\
x \vdash y
\end{array}
$$

Canonical cut elimination

- Subcase (b2): a_u principal in other rule. Then, a_u is in display, and hence:

$$
\dfrac{\begin{array}{c} \vdots\, \pi_1 \\ x \vdash a \end{array} \qquad \begin{array}{c} \vdots\, \pi_2' \\ a_u \vdash y' \\ \vdots\, \pi_2 \\ a \vdash y \end{array}}{x \vdash y}
\quad \rightsquigarrow \quad
\begin{array}{c} \dfrac{\begin{array}{c} \vdots\, \pi_1 \\ x \vdash a \end{array} \qquad \begin{array}{c} \vdots\, \pi_2' \\ a_u \vdash y' \end{array}}{x \vdash y'} \\ \vdots\, \pi_2[x/a] \\ x \vdash y \end{array}
$$

Canonical cut elimination

- Subcase (b2): a_u principal in other rule. Then, a_u is in display, and hence:

$$
\dfrac{\dfrac{\vdots\ \pi_1}{x \vdash a} \qquad \dfrac{\dfrac{\vdots\ \pi_2'}{a_u \vdash y'}}{\dfrac{\vdots\ \pi_2}{a \vdash y}}}{x \vdash y}
\quad \rightsquigarrow \quad
\dfrac{\dfrac{\dfrac{\vdots\ \pi_1}{x \vdash a} \qquad \dfrac{\vdots\ \pi_2'}{a_u \vdash y'}}{x \vdash y'}}{\dfrac{\vdots\ \pi_2[x/a]}{x \vdash y}}
$$

- Subcase (b3): a_u parametric. Then:

$$
\dfrac{\dfrac{\vdots\ \pi_1}{x \vdash a} \qquad \dfrac{\dfrac{\vdots\ \pi_2'}{(x' \vdash y')[a_u]^{pre}}}{\dfrac{\vdots\ \pi_2}{a \vdash y}}}{x \vdash y}
\quad \rightsquigarrow \quad
\dfrac{\dfrac{\vdots\ \pi_2'}{(x' \vdash y')[x/a_u^{pre}]}}{\dfrac{\vdots\ \pi_2[x/a_u^{pre}]}{x \vdash y}}
$$